Superconducting Cosmic Strings

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Abstract

It is well known that certain spontaneously broken gauge theories give rise to stable strings or vortex lines. In this dissertation we shall review the mechanisms of their formation in field theories, together with the topological constraints on the manifold of degenerate vacua. We shall then turn towards their possible role in the Universe. In the final two sections we shall discuss an interesting possibility of having a cosmic string which behaves like a superconducting wire with bosonic charge carriers.

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1 Introduction to Topological Defects

In the following three subsections we shall discuss the basic mechanisms of the formation of point-like (monopoles), line-like (cosmic strings) and wall-like (domain walls) defects predicted in certain field theories. We shall focus our attention to local cosmic strings (line-like defects) as our main aim is to discuss the possibility of superconductivity in these line-like defects. Also they might have a role to play in the formation of large scale structure in the universe. We shall be discussing their formation purely at the field theory level without any mention of cosmology. Their cosmological consequences will be discussed in the next section.

1.1 Domain Walls

Let us begin with a very simple model which exhibits the formation of domain walls. Consider a real scalar field ϕ with the Lagrangian

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{1}$$

where the potential is of the form (see Figure 1)

$$V(\phi) = \frac{1}{4}\lambda(\phi^2 - v^2)^2 \tag{2}$$

Obviously, the Lagrangian has the discrete symmetry $\phi \to -\phi$. However, as can be seen from the figure, the only value of ϕ which respects this symmetry is $\phi = 0$. But $\phi = 0$ is a local maxima. The field will relax to lower energy values of either v or $-v$, which are the two degenerate minima of the potential. One can ask the question what will be the equilibrium field configuration, or the vacuum expectation value of ϕ. Either value will suffice. It is possible for the field to take one value $<\phi>= v$ in one region of space and the other value $<\phi>= -v$ in another part of space. If it happens to be the case, then somewhere in the middle $<\phi>$ must go to zero, as one cannot go continuously from $<\phi>= v$ to $<\phi>= -v$. If $<\phi>= 0$ then it means that the state of higher energy is trapped in between the two regions. It should be noted that this region must either be infinite or closed, otherwise it would be possible to continuously go over this region. Such infinite walls of trapped energy density are called Domain Walls (see Figure 2).

With the above Lagrangian, Euler-Lagrangian equations give the equation of motion for ϕ,

$$\Box^2\phi + \lambda\phi(\phi^2 - v^2) = 0 \tag{3}$$

Chosing the boundary conditions to be $< \phi > \to +v$ as $z \to \infty$ and $< \phi > \to -v$ as $z \to -\infty$, we get the lowest-energy static solution of this equation as

$$\phi_0(z) = v \tanh(z/\delta) \tag{4}$$

where

$$\delta \equiv (\frac{\lambda}{2})^{-1/2}\frac{1}{v} \tag{5}$$

can be regarded as the thickness of the domain wall. This solution is plotted in Figure 3a. The stress tensor for the domain wall is obtained from the expression for the stress-energy tensor for a scalar field

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - Lg_{\mu\nu} \tag{6}$$

Using the wall solution we obtain

$$T_{\mu\nu} = \frac{\lambda}{2}v^4 \cosh^{-4}(z/\delta)\ diag(1,1,1,0) \tag{7}$$

This shows that the x- and y-components of the pressure are equal to minus the energy density, whereas the x-component vanishes. The energy density associated with the wall, i.e., T^0_0, as a function of z, is a bell-shaped function and plotted in Figure 3b. Using this we can calculate the surface energy density of the domain wall as

$$\sigma = \int_{-\infty}^{+\infty} T^0_0 dz = \frac{2\sqrt{2}}{3}(\frac{\lambda}{2})^{1/2}v^3 \tag{8}$$

which is obviously equal to the integrated transverse components of the stress, that is, the surface tension in the wall is precisely equal to the surface energy density.

For the stress tensor

$$T_{\mu\nu} = diag(\rho, -p_1, -p_2, -p_3), \tag{9}$$

the Newtonian limit of Poisson's equation is

$$\nabla^2\Phi = 4\pi G(\rho + p_1 + p_2 + p_3) \tag{10}$$

For the domain wall lying along z-plane, $p_3 = 0$ and $p_1 = p_2 = -\rho$, and so we get

$$\nabla^2\Phi = -4\pi G\rho \tag{11}$$

This negative sign has the important effect that the gravitational field of the wall is repulsive, i.e., gravitational test particles are repelled by an infinite domain wall.

1.2 Strings or Vortex Lines

In the following sections we shall start with a discussion of local strings arising from spontaneous breaking of a $U(1)$ gauge symmetry. Then we shall discuss the possibility of having string solutions for the case of symmetry breaking of higher symmetry groups. In the third section we shall talk about global strings and how they differ in their properties from local strings.

From the point of view of particle physicists, cosmic strings are one dimensional topological defects arising in some grand unified gauge theories. For astronomers they can also be viewed as lines of trapped energy density which is frozen in during the very early Universe.

1.2.1 Local strings

The simplest model which admits a string solution is that of a complex scalar field with the following Lagrangian, invariant under U(1) gauge transformations.

$$L = D_\mu\phi D^\mu\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(\phi) \tag{12}$$

where

$$V(\phi) = \frac{1}{2}\lambda(\phi^\dagger\phi - \eta^2)^2 \tag{13}$$

with

$$D_\mu\phi = (\partial_\mu - \iota g A_\mu)\phi \tag{14}$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{15}$$

The potential in the above equation is the famous Mexican hat potential plotted in Figure 4.

The above Lagrangian is invariant under $\phi \rightarrow \phi e^{\iota\alpha(x)}$ The ground state is $<\phi>=\eta e^{\iota\theta(x)}$. As θ is x_μ dependent, it can take arbitrary values at different space-time points. But ϕ is single valued, and so the total change in θ around any closed loop must be $2\pi n$, where n is an integer. Suppose we find a closed lŏop with $n = 1$. Imagine shrinking it to a point. If no zeros of ϕ are encountered, n cannot discontinuously change from $n = 1$ to $n = 0$. Somewhere within the loop ϕ must go to zero, which means that the phase θ is ill-defined at that point (point A in the figure 5). In fact ϕ should be zero not only at A but on points above and bellow it (like B, C,.. and X, Y,..), otherwise it would be possible to deform the loop a little bit and then shrink it over $<\phi>=0$. This line of trapped energy density is called a String or Vortex line (see Figure 5). Because of the reasons given above, they must either be in the form of infinite string or closed loop. The radius of the string core is approximately equal to the Compton wavelengths of the Higgs and vector mesons, i.e., $\delta_\phi \sim m_\phi^{-1}$ and $\delta_A \sim m_A^{-1}$.

Let us now calculate the magnetic flux of such an string. At large distance from the string, the scalar field takes the form

$$\phi(r) = \eta e^{\iota n\vartheta} \tag{16}$$

To make the energy per unit length of a string at large distance from the string finite we must have $D_\mu\phi = 0$. This is because the energy density, for a static configuration,0 is given by the integral of Hamiltonian

$$\rho = \int d^2x[\frac{1}{2}\vec{D}\phi^\dagger \cdot \vec{D}\phi + V(\phi)] \tag{17}$$

The second term is zero at large distances (see the form of ϕ at large distances). The first term must also vanish to make the whole thing finite. The requirement $D_\mu\phi = 0$ at large distances from the string thus gives

$$A_\mu = \frac{-\iota}{g}\partial_\mu \ln(\frac{\phi}{\eta}) = \frac{n}{g}\partial_\mu\vartheta \tag{18}$$

The form of A_μ is such that $F_{\mu\nu} = 0$, so the energy density vanishes outside the string core. Note that it is essential to have a gauge field to get a finite energy solution at infinity. Also note that at the core of the string both the fields ϕ and A_μ vanish. Since $\vec{B} = \vec{\nabla} \times \vec{A}$, from stokes's theorem we get

$$\int \vec{B}\cdot d\vec{S} = \oint \vec{A}\cdot d\vec{l} = \frac{-\iota}{g}\int \nabla(\ln(\frac{\phi}{\eta}))\cdot dl = \frac{2\pi n}{g} \tag{19}$$

This equation shows that string carries n units of magnetic flux $2\pi/g$. String can therefore be thought of as a relativistic analogue of quantized flux tubes in an ordinary superconductor.

Finally we calculate the energy-momentum tensor for a cosmic string. Ignoring their internal structure as compared to their cosmological dimension, we can average over the cross section. For a static straight string lying along the z-axis we define

$$\tilde{T}^{\nu}_{\mu} = \delta(x)\delta(y)\int T^{\nu}_{\mu}\,dxdy \tag{20}$$

The string solution is invariant under Lorentz boosts in the z-direction (and in time) and thus $\tilde{T}^0_0 = \tilde{T}^3_3$ with all other off-diagonal components equal to zero. To show that the remaining diagonal components are also zero we use the conservation law $T^j_{i,j} = 0$ to get $(i, j = 1, 2)$

$$\int T^j_{i,j}\, x^k\, dxdy = 0 \tag{21}$$

Integrating by parts we get

$$\int T^k_i\, dxdy = 0 \tag{22}$$

Thus, the energy-momentum tensor of the string is given by

$$\tilde{T}^{\nu}_{\mu} = \mu\ diag(1, 0, 0, -1)\delta(x)\delta(y) \tag{23}$$

Note that the pressure is negative–it is a string tension. Thus the tension along the string is equal to the energy density.

Here again we use the Poisson's equation to get

$$\vec{\nabla}^2\Phi = 0 \tag{24}$$

which suggests that the space outside of an infinite string is flat. We shall discuss this point further in the section on topological defects in the Universe.

1.2.2 More complicated strings

In the previous section we discussed the formation of vortex line only for the case when $U(1)$ symmetry is spontaneously broken. However the formation of strings is possible under fairly general assumptions about the gauge groups involved. To this end we would like to borrow a theorem from theory of homotopy groups.

A Useful Theorem:
Consider a chain of symmetry breakings $G \to H \to H' \cdots$. Let M be the manifold of degenerate vacua. This means that we have a symmetry group G which is spontaneously broken by a Higgs field ϕ to a subgroup H. So we can write $H = \{g \in G : g\phi = \phi\}$. Thus if we have one point η on the minimum of M we can obtain any other point by applying elements of G to η. Note that M cannot be the entire G because if we apply two different elements g and g' we may obtain the same point on M i.e., $g\phi = g'\phi$. This is true iff $g^{-1}g' \in H$ so that g and g' are related by left multiplication in H. Hence we can identify M with G/H i.e., $M = G/H$, where equality sign represents isomorphism. Then we have the following theorem:
If

$$\pi_n(G) = \pi_{n-1}(G) = I,$$

then

$$\pi_n(M) = \pi_{n-1}(H) \tag{25}$$

For example if

$$\pi_1(G) = \pi_0(G) = I$$

then we get

$$\pi_1(M) = \pi_0(H) \tag{26}$$

This shows that strings can be formed in a phase transition $G \to H$ if $\pi_0(H) \neq I$.

Hence we can classify the defects in the following way:
Defect———————classified according to
Wall——————— $\pi_0(M)$
String——————— $\pi_1(M)$
Monopole——————— $\pi_2(M)$

So for the case of Domain walls, $\pi_0(M)$ counts the disconnected pieces in the manifold of degenerate vacua. We shall discuss these matters a little further in the next chapter when we come to the cosmological consequences of these strings.

1.2.3 Global strings

The topological reasons for the formation of global strings are the same as for the formation of local strings. We just put $A_\mu = 0$ in the Lagrangian for a complex scalar field. The only difference comes from the fact that now we don't have a compensating field to make the covariant derivative of ϕ equal to zero, at large distances from the string. The Lagrangian now becomes

$$L = \frac{1}{2}\partial^\mu \phi^\dagger \partial_\mu \phi - \frac{\lambda}{4}(\phi^\dagger \phi - v^2) \tag{27}$$

It has a $U(1)$ symmetry $\phi \to e^{i\alpha}\phi$ where α is a constant. Here again $<\phi>= 0$ is an higher energy state and will relax to lower energy states. The vacuum expectation value of ϕ is $<\phi>= ve^{i\theta(x)}$. As before v i.e., $|<\phi>|$ is fixed by the model, but $\theta(x)$ which is position dependent, can be anything. This fact again gives rise to the formation of line-like defects called global strings. A straight global string along the $+z$-axis is the static configuration

$$\phi(x) = vf(r)e^{i\theta} \tag{28}$$

where θ and r are the polar coordinates in the plane perpendicular to z-axis. The function f obeys the boundary conditions
$f \to 1$ as $r \to \infty$
$f \to 0$ as $r \to 0$
and is such that the equation of motion

$$\partial_\mu \partial^\mu \phi = \lambda(\phi^\dagger \phi - v^2)\phi \tag{29}$$

is obeyed. So at large distances from the string we can write

$$\phi(x) = ve^{ia(x)/v} \tag{30}$$

where a is the Nambu-Godstone boson associated with the spontaneous breaking of the global $U(1)$ symmetry.

We can calculate the energy per unit length of the straight string as

$$\rho = \int d^2x[\frac{1}{2}\vec{\partial}\phi^\dagger \cdot \vec{\partial}\phi + V(\phi)] \tag{31}$$

At large distances from the string core, the second term does not contribute and we get, using eq.28 for large r

$$\rho = \pi v^2 \int_\delta^{\mathrm{L}} \frac{dr}{r} = \pi v^2 \ln(L/\delta) \tag{32}$$

as $\vec{\partial} = \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}$, and where L is a large distance cutoff. δ is the 'core' of the string, inside which the limit $f50z \to 1$ is not valid.

For a straight global string along the z-axis, the energy momentum tensor is (from equation 6)

$$T_{00}(\mathbf{x}) = v^2/2r^2$$

$$T_{0i} = 0$$

$$\mathbf{T}(\mathbf{x}) = \frac{v^2}{2r^2}(\hat{\mathbf{z}}\hat{\mathbf{z}} + \hat{\theta}\hat{\theta} - \hat{\mathbf{r}}\hat{\mathbf{r}}) \tag{33}$$

outside the string core. Einstein's equations (in the linear approximation) then give the following form of the metric

$$ds^2 = (1 - 4\pi Gv^2 \ln(r/\delta))(-dt^2 + dz^2) + dr^2 + (1 - 8\pi Gv^2(\ln(r/\delta) + c))r^2 d\theta^2 \tag{34}$$

where c is the energy per unit length of the string core in units of πv^2. The important difference from local string case is that the angle deficit

$$\alpha(r) = 8\pi^2 Gv^2(\ln(r/\delta) + c) \tag{35}$$

increases logarithmically with the distance r to the string center. Also, from Poisson's equation, we can see that the global string produces a repulsive gravitational potential. At large distances from the string

$$\ddot{r} = \frac{2\pi Gv^2}{r} \tag{36}$$

Global strings may arise whenever a global symmetry is broken. One of the most common example is the breaking of global $U(1)$ axial symmetry of QCD Lagrangian (the so called Peccei-Quinn or PQ symmetry). The Goldstone bosons arising due to the breaking of this axial symmetry are called axions and the corresponding strings are known as axionic strings.

1.3 Monopoles

As discussed in the previous section, the formation of topological defects is dictated by the topology of the gauge groups involved. In particular we saw that strings can be formed whenever $\pi_0(H)$ is nontrivial. The next higher possibility is to have $\pi_1(H)$ non trivial. The defects which originate in such a symmetry breaking are called monopoles (see Figure 6) and are point-like defects. It should be kept in mind that they are not the Dirac monopoles of classical electricity and magnetism. However, as we shall see, they can be thought of as quantum analogues of the Dirac monopoles. The simplest example of a gauge theory which predicts monopole solutions is the $SU(2)$ model of Georgi and Glashow. It contains a massless photon and two charged weak bosons that acquire mass from a Higgs isotriplet. It is ruled out by the experimental discovery of neutral currents. As it illustrates the formation of monopoles rather well, we shall consider this model. It has the following Lagrangian with an isotriplet of Higgs scalars ϕ^i $(i = 1, 2, 3)$

$$L = -\frac{1}{4}F_i^{\mu\nu}F_{\mu\nu}^i + \frac{1}{2}(D^\mu\phi^i)(D_\mu\phi^i) - V(\phi) \tag{37}$$

where the field strength tensor is

$$F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i - e\epsilon^{ijk}A_\mu^j A_\nu^k \tag{38}$$

and the covariant derivative is

$$D_\mu\phi^i = \partial_\mu\phi^i - e\epsilon^{ijk}A_\mu^j\phi^k \tag{39}$$

The Higgs potential has again the usual structure

$$V(\phi) = \frac{1}{4}\lambda(\phi^i\phi^i - v^2)^2 \tag{40}$$

The values of ϕ which minimize this potential constitute a sphere in the isospin space and hence are related to each other by an internal $SO(3)$ transformation. However we if choose the ground state for the Higgs to be

$$\phi = (0, 0, v) \tag{41}$$

then this symmetry is spontaneously broken, and $SO(3) \rightarrow SO(2) = U(1)$ because ϕ is still invariant under $SO(2)$ rotations about the third axis. Once again, the finite-energy solution is obtained by requiring that as $r \rightarrow \infty$ and $\phi \rightarrow$ constant and $D_\mu\phi \rightarrow 0$. The simplest possibility is to take (keeping in mind the vacuum expectation value of ϕ),

$$\phi^i(r) \rightarrow v,\ \phi^{1,2}(r) \rightarrow 0 \tag{42}$$

in all directions in space, as in Figure 7. Another nontrivial and interesting possibility is to take

$$\phi^i(r) \rightarrow vr^i$$

where r^i is a unit vector in the direction $i\,(i = 1, 2, 3)$ in coordinate space. This means that we have correlated the "direction" of ϕ in the internal space with the direction in coordinate space. Such a solution is sometimes called the "hedgehog" solution (see Figure 7). Using the condition on covariant derivative we get for A

$$A_i^j(r) \rightarrow \epsilon_{ijk} \frac{r_k}{er^2} \tag{43}$$

With $A_0^i = 0$ we obtain $E_i \rightarrow 0$ and $B_i \rightarrow \frac{r_i}{er^3}$ which is a radial magnetic field, like that of a Dirac monopole. The total magnetic flux through the sphere at infinity is

$$\Phi = 4\pi r^2 B = \frac{4\pi}{e} \tag{44}$$

which is the analogue of famous Dirac quantization condition for a magnetic monopole.

There are many other interesting properties of these monopole solutions but we shall not go into the details of them.

2 Topological Defects in the Universe

In the last section we only discussed the possibility of having topological defects in certain models of field theory. We shall now show that such defects may arise during the evolution of our Universe. Before discussing this we must clarify what we mean by phase transitions in the context of field theory and cosmology. After having a brief discussion of this we will go on to discuss domain walls and strings, giving some specific examples from grand unification models.

2.1 Cosmological Phase Transition

Naively speaking, a phase transition is the transition of a system from one physical state to another state with different physical properties, as the system cools down (or heats up) a particular temperature T_c. However we shall be interested in phase transitions in the context of field theory. Although it is possible to develop the arguments without any assumptions about symmetry groups involved or the Higgs structure, we shall consider the simplest case of $U(1)$ group.

Consider the following form of the Higgs potential for a complex scalar field ϕ

$$V(\phi) = \frac{1}{2}\lambda(\phi^\dagger\phi - \eta^2)^2 \tag{45}$$

with $\lambda > 0$. The $U(1)$ symmetry is the symmetry of phase transformations, $\phi \rightarrow e^{\iota\alpha}\phi$. The minima of the potential is at nonzero values of ϕ. So the symmetry is spontaneously broken and ϕ acquires a vacuum expectation value

$$<\phi>= \eta e^{\iota\theta} \tag{46}$$

We thus have a manifold, M, of degenerate vacuum states corresponding to different values of θ. In fact M is a circle of radius η.

At finite temperatures the effective potential takes the form

$$V_T(\phi) = AT^2\phi^\dagger\phi + V(\phi) \tag{47}$$

We shall assume that $A > 0$. This means that the effective mass of the Higgs field is temperature dependent

$$m^2(T) = AT^2 - \lambda\eta^2 \tag{48}$$

which is zero at $T = T_c$, where

$$T_c = (\lambda/A)^{1/2}\eta \tag{49}$$

Unless λ is very small, we have $T_c \sim \eta$. For $T > T_c$, $m^2(T)$ is positive, the minimum of $V(\phi)$ is at $\phi = 0$, and so the expectation value of ϕ vanishes and the symmetry is restored. The situation is depicted in Figure 8.

We can generalize the above discussion to the case of symmetry breaking $G \rightarrow H$, where H includes all elements of G which leave the vacuum expectation value $< \phi >$ invariant. The manifold of vacuum states, M, can be identified with the coset space G/H.

We can now recast our discussion of topological defects in Section 1 in a form suitable for discussion in the context of grand unification and Cosmology. We study the chain of symmetry breakings

$$G \rightarrow H \rightarrow \cdots \rightarrow SU(3)_c \times SU(2)_L \times U(1)_Y \rightarrow SU(3)_c \times U(1)_{em} \tag{50}$$

where each symmetry breaking occurs at a particular temperature (or energy), due the vacuum expectation value of some Higgs field.

2.2 The Kibble Mechanism

We have shown in the last section that the formation of toplogical defects is possible if we impose conditions on the topology of M-the manifold of degenrate vacuum states. In this section we shall show that their formation is inevitable if we consider certain grand unification model in the context of cosmology. The mechanism which justifies this is called Kibble mechanis.

First we introduce a correlation length ξ, which must be the particle horizon in the context of cosmology. Particle horizon means the maximum distance over which a massless particle could have propagated since the time of big bang. The fact that the horzon distance is finite implies that at the time of phase transition, the Higgs field must be uncorrelated on scales greater than ξ.

During a SSB phase transition, some Higgs field acquire a VEV. Because of the existence of the particle horizon, $< \phi >$ cannot be correlated on scales larger than ξ. Thus topological defects will necessarily be produced, with an abundance of order one per horizon volume. As they are stable, once formed, they are "frozen in" as permanent defects.

2.3 Formation of Domain Walls

We have already discussed the formation of Domain walls in chapter 1 and we shall not repeat that discussion here. We just want to add two things.

It is possible in certain Grand Unification models to form Domain Walls at early times in the evolution of the Universe. At that time the horizon length was much less than the present. These walls, once formed, are stable objects. As the Universe cools and expand, they acquire a permanent status.

Recall our expression (equation 8) for the energy density of a domain wall.

$$\sigma = \frac{2}{3}(\frac{\lambda}{2})^{1/2}v^3 \sim M_X^3 \tag{51}$$

For walls of cosmic sizes we can ignore their thickness. The mass of a domain wall of dimension R_H would be $M_W \simeq \sigma R_H^2$, and mass density $\rho \simeq \sigma / R_H$, where $R_H \simeq H_0^{-1}$ is the distance to the Hubble horizon. To keep $\rho < \rho_c$, requires

$$\sigma < R_H \rho_c = \frac{3H_0}{8\pi G_N} \tag{52}$$

Or, using $\sigma \sim M_X^3$ and putting values for the parameters, we get $M_X \leq 10^{-2} GeV$. This shows that any model which predicts domain wall formation even at energy scales much below the electroweak breaking scale, is ruled out by many orders of magnitude.

One possible way out of this problem is to assume that domains wall were formed before the inflationary period (if there was any) so that they inflated out of our horizon. However they may still have a role to play in the formation of large scale structure. For example the recent observations that galaxies appear to be distributed not randomly but on the surfaces of large bubbles, might be explained on the basis of an underlying domain structure.

Now we would like to point out certain specific GUT models which predict the formation of domain walls.

1. $SU(5)$ invariant Lagrangian with discrete symmetry $\phi \rightarrow -\phi$ gives rise to a potential with two distinct vacuum states and hence to Domain walls.

2. $SO(10)$ model with discrete symmetry $\phi \rightarrow -\phi$ may also give rise to domain walls if the symmetry breaking pattern is $SO(10) \xrightarrow{16} SU(5) \xrightarrow{45} SU(3) \times SU(2) \times U(1)$

2.4 Formation of Cosmic Strings

To have a rough idea of objects we are going to discuss, let us add some typical dimensions.

Mass per unit length $\sim 10^{15}$ tons/cm

Width $\sim 10^{-22}\, r_H$, where r_H is the radius of Hydrogen atom.

Formation time $\sim 10^{-35}$ s after the big bang.

Recall that the mass per unit length of a cosmic string is $\mu \sim v^2$. This means that the mass of an infinite string (i.e., of the size of present horizon) would be

$$M = \mu R_H = \frac{\mu}{H_0} \simeq 10^{-15} (\frac{v}{GeV})^2 M_\odot \tag{53}$$

Putting the values for the parameters, i.e., $v \simeq M_X \simeq 10^{15}$ GeV, a convenient value for grand unification, we get $M \simeq 10^{15} M_\odot$, which is of the order of the mass of a galaxy cluster. Similarly a loop of radius L would have a mass $M \simeq \mu L \simeq v^2 L$.

It is possible to have a system of many strings. The typical scale of the system is $\sim \xi$, the correlation length. If there is one string segment per volume $\sim \xi$, then the mass density of such a system would be $\rho_S \sim \mu \xi^{-2}$.

Before proceeding further let us add another important fact about strings. The metric near the string has the form

$$ds^2 = dt^2 - dz^2 - dr^2 - (1 - 4G\mu)^2 r^2 d\phi^2 \tag{54}$$

A simple coordinate transformation $\phi' = (1 - 4G\mu)\phi$ brings it to a locally Minkowskian form. But the important point to note is that the range of values of ϕ' is from 0 to $(1 - 4G\mu)2\pi$ and not the usual range 0 to 2π. Such a space is called a conical space, that is, a flat space with a wedge of angular size $8\pi G\mu$ taken out and two faces of the wedge identified (see Figure 9). In the coordinates (t, z, r, ϕ'), the geodesics are straight lines and particles at rest with respect to string will remain at rest, i.e., will not experience any gravitational attraction.

However it should be kept in mind that the space around a string is only locally flat. Particles passing on opposite sides of the string are deflected towards one another by an angle

$$\Delta\phi = 8\pi G\mu \tag{55}$$

This means that matter passing on the opposite sides of the string is attracted towards each other. Due to this mechanism, long string segments can give rise to large scale structure apparent in the universe.

Another mechanism by which strings can seed structure relies on the fact that at large distances from a loop, the gravitational field is like that of a point mass. This means that matter may start accreting onto loops, giving rise to objects like galaxies and clusters.

Finally we point out some grand unification models which predict strings.

1. The simplest model which predicts global cosmic strings is the well known $SU(5)$ model which has a $U(1)_{B-L}$ global symmetry.

2. It is possible to form topologically stable strings in $SO(10)$ model with the following chain of symmetry breakings $SO(10) \overset{M_G}{\to} SU(5) \times Z_2 \overset{M_X}{\to} SU(3) \times U(1)_{em} \times Z_2$

3 Superconducting Cosmic Strings

3.1 Introduction

Upto now we were discussing the simplest cases of string formation. In the following sections we shall show that certain models, with suitable choice of the Higgs potential parameters, predict the formation of a superconducting cosmic string. Although it is possible to have superconductivity due to scalars, fermions or charged vector bosons, we shall discuss only the first case, i.e., superconductivity due to bosonic (spin-0) charge carriers. We shall show that the essential arguments for superconductivity are particularly clear for this case. For the other two cases we refer the reader to the literature suggested in the last section.

3.2 Bosonic Charge Carriers

Let us first fix the notation. We are going to discuss a $U(1) \times U(\tilde{1})$ gauge invariant Lagrangian with scalar fields σ and ϕ and two vector fields A and B. The respective gauge couplings are e and g.

We shall consider the situation where the $U(\tilde{1})$ symmetry is spontaneously broken due to the expectation value of the Higgs field ϕ whereas the $U(1)$ symmetry of electromagnetism remains intact.

Consider the Lagrangian

$$\mathrm{L}_{kin} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{4}R_{\mu\nu}^2 + D_\mu\sigma^\star D^\mu\sigma + D_\mu\phi^\star D^\mu\phi \tag{56}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{57}$$

$$R_{\mu\nu} = \partial_\mu R_\nu - \partial_\nu R_\mu \tag{58}$$

and the covariant derivatives are

$$D_\mu\sigma = (\partial_\mu + \iota e A_\mu)\sigma \tag{59}$$

$$D_\mu\phi = (\partial_\mu + \iota g R_\mu)\phi \tag{60}$$

and $V(\phi)$ is the quartic gauge invariant potential of the form

$$V(\sigma, \phi) = \frac{1}{8}\lambda(|\phi|^2 - \mu^2)^2 + \frac{1}{4}\tilde{\lambda}|\sigma|^2 + f|\sigma|^2|\phi|^2 - m^2|\sigma|^2 \tag{61}$$

where we add the following constraints whose importance will become clear very soon.

$$\mu^2, m^2 > 0, \ \ f|\phi|^2 - m^2 > 0$$

With these conditions, it is easy to show that the minima of the potential is at $< \sigma >= 0$, $| < \phi > | = \mu$. This means that electromagnetism is unbroken but the $U(\tilde{1})$ symmetry or R is spontaneously broken. The breakdown of R leads to the existence of string solution.

In the vortex field, ϕ is independent of two coordinates, say z and t: ϕ vanishes at $x = y = 0$. The important point to note is that with the particular choice of the parameters the potential energy favors $< \sigma >= 0$ in the core of the string where $\phi = 0$, ($< \sigma > \neq 0$ only in the vacuum). The kinetic energy tends to resist this as σ must vanish at large distances from the string.

We must explore the balance between kinetic energy and potential energy. There certainly exist a solution with both σ and A_μ equal to zero. The equation for small fluctuations in σ around a ϕ background is

$$\ddot{\sigma} - \nabla^2\sigma + (f|\phi|^2 - m^2)\sigma = 0 \tag{62}$$

Assuming that the solution can be written in the form (explained later)

$$\sigma(x, y, z, t) = e^{-\iota\omega t}\sigma_0(x, y)$$

we get an equation for σ_0

$$(-\frac{d^2}{dx^2} - \frac{d^2}{dy^2})\sigma_0 + V(r)\sigma_0 = \omega^2\sigma_0 \tag{63}$$

where $V(r) = f|\phi|^2 - m^2$. This is a two dimensional Schrodinger equation with potential $V(r)$.

Let us look at the behavior of the potential closely. It is attractive near $r = 0$, $V(0) = -m^2$, and increases monotonically to $V(\infty) = f\mu^2 - m^2$ as r increases to large values. What is more important is that there is an allowed range of parameters in which the potential is negative definite. For example if we take $m^2 = f\mu^2$ then this condition can be satisfied. It is known that in two dimensions the Schrodinger equation with a negative definite potential always has a bound state, so there is certainly a bound state if $m^2 = f\mu^2$.

By continuity there is also a bound state solution if $m^2 < f\mu^2$. We shall see that the interesting effects arise in this range.

As discussed earlier, potential energy favors $\sigma \neq 0$ in the string. But this means, roughly, that electromagnetism is spontaneously broken and there will be Goldstone bosons carrying charges up and down the string.

Let $\sigma_0(x, y)$ be the value of the σ field that minimizes the energy. Then $e^{i\vartheta}\sigma_0$ also minimizes the energy for any real constant ϑ. We therefore take the following ansatz for the σ field, motivated by the fact that the string carries massless Goldstone bosons in the z, t direction

$$\sigma(x, y, z, t) = e^{i\vartheta(z,t)}\sigma_0(x, y) \tag{64}$$

where $\vartheta(z, t)$ is an arbitrarily slowly varying function. These excitations will be responsible for making the string a superconducting wire.

Assuming that A_μ is a slowly varying function on the scale of the string we can set $A_\mu(x, y, z, t) = A_\mu(0, 0, z, t) \equiv A_\mu(z, t)$ whenever $\sigma \neq 0$ .Using the above form for ϑ we get the following form for the Lagrangian

$$L = |\sigma_0|^2[(\partial_0\vartheta(z, t) + eA_0(z, t)^2 - (\partial_z\vartheta(z, t) + eA_z(z, t)^2] \tag{65}$$

which immediately gives the effective action for ϑ

$$I_\vartheta = K\int dzdt[(\partial_0\vartheta(z, t) + eA_0(z, t)^2 - (\partial_z\vartheta(z, t) + eA_z(z, t)^2] \tag{66}$$

where

$$K = \int dxdy|\sigma_0(x, y)|^2 \tag{67}$$

To describe the long-wavelength interactions of strings with the electromagnetic fields we add to it the standard electromagnetic action to get

$$I = I_A + I_\vartheta = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} + K\int dzdt(\partial_i\vartheta + eA_i)^2 \tag{68}$$

Now we can compute the electromagnetic current as $J_i = -\delta I_\theta/\delta A_i$. This gives

$$J_i(z, t) = 2Ke(\partial_i\vartheta + eA_i) \tag{69}$$

Given this expression for the current we shall show that there are current carrying states of the string which do not relax.

First note that it is possible to define a topological invariant

$$N = \frac{1}{2\pi} \oint dl \frac{d\vartheta}{dl} \tag{70}$$

where l is a parameter along the string. This essentially gives the winding number of the ϑ field. It is obvious that total change in ϑ around any closed loop should be 2π times an integer. Thus in the above equation N should be an integer.

It should be kept in mind that the reasons of N being conserved are not topological. It is only energetically favored to have N fixed. In the following we shall try to make it clear. Consider a loop of string with some fixed N. As ϑ is the phase of the σ field, it is ill-defined when $\sigma = 0$. So N can change if σ passes through zero at some point on the string at some time. But this situation is not energetically favored. Hence N will remain fixed. Only when the current flowing in the string becomes so large that its energy content is comparable to the energy needed to set $\sigma = 0$ do the process in which N changes occur.

Let us now compute the current carried by a string (not necessarily circular) of circumference $2\pi R$ in its lowest energy state of fixed N. The equation for the vector potential can be derived from the Lagrangian given above. We get

$$\nabla^2 A_i - \partial_i(\nabla \cdot A) = J_i \tag{71}$$

where we have used the previously derived value of current to get the right hand side of this equation. It is not an easy matter to solve this equation in general. However in the special choice of gauge $\nabla \cdot A = 0$ we can get the solution very easily

$$A_i(x) = -\frac{1}{4\pi} \oint dl \frac{1}{|x - x(l)|} J_i(x(l)). \tag{72}$$

It should be noted that the above integral diverges logarithmically when l is such that $x = x(l)$. This singularity arises because we are assuming that the string is of zero thickness. If we take into account the finite thickness of the string, this singularity becomes mild. Then we can break the integral into three pieces (one finite and two infinite). Ignoring the finite part we get, on dimensional grounds

$$A_i(x(l)) = -\frac{\ln(\Lambda R)}{2\pi} J_i(x(l)). \tag{73}$$

For simplicity we shall denote J_i and A_i along the string as J and A respectively. The above equation then says $A = -\ln(\Lambda R)\, J/2\pi$. By current conservation J is a constant along the string and so is A. This gives

$$J = 2Ke(\frac{d\vartheta}{dl} + eA) \tag{74}$$

Since J and A are constant along the string in this gauge, the same is true for $d\vartheta/dl$. Since

$$N = \frac{1}{2\pi} \oint dl \frac{d\vartheta}{dl} \tag{75}$$

we can calculate the value of this constant

$$\frac{d\vartheta}{dl} = \frac{N}{R} \tag{76}$$

Combining all the above equations we finally get

$$J = \frac{2Ke}{1 + Ke^2 \ln(\Lambda R)/\pi} \frac{N}{R} \tag{77}$$

For $K >> 1$ we get,

$$J = \frac{2\pi}{e\, \ln(\Lambda R)} \frac{N}{R} \tag{78}$$

This is the cental result of this dissertation. It shows that in the situation where N is fixed (energetically favourable condition), there are currents on the string which do not decay in time.

Before finishing this first section let us calculate one more thing which is the total change in the average current on the string in an arbitrary time dependent process. We have

$$\oint J dl = 2Ke \oint dl(\frac{d\vartheta}{dl} + eA) = 4\pi KeN + 2Ke^2 \oint dl A_i \frac{dx^i}{dl} \tag{79}$$

Here N is time independent and

$$\oint dl A_i \frac{dx^i}{dl} = \Phi \tag{80}$$

Φ being the magnetic flux through any surface spanning the string. So

$$\frac{d}{dt} \oint J dl = 2Ke^2 \frac{d\Phi}{dt} \tag{81}$$

One might expect that by Meissner effect, magnetic flux lines could not cross the string and $d\Phi/dt$ would vanish. However this is not the case, essentially because we are dealing with a thin superconductor whose dimensions are much less than the characteristic magnetic penetration length.

4 Strings in magnetic fields

In the last few sections we discussed the formation and properties of the domain walls, cosmic strings, monopoles and superconducting cosmic strings. Now we shall discuss the interactions of the superconducting variety with magnetic fields.

Let us consider scattering of light by the string. To make the arguments simple we shall assume that the light is incident at 90^o to the string, with the polarization such that the electric field vector is parallel to the string which is lying along the z-axis.

We choose a gauge with $A_t = A_x = A_y = 0$, and since the problem is z-independent we can take $A_z = A(x,y,t)$. The equation of motion then becomes, with $A(x,y,t) = A(x,y)e^{-\iota\omega t}$ (starting from equation 68)

$$(-\nabla^2 + 2Ke^2\delta^2(x))A(x,y) = \omega^2 A(x,y) \tag{82}$$

This is a Schrodinger equation for scattering from a delta function potential. The scattering solution obeys

$$A(\vec{x}) = e^{\iota\vec{k}\cdot\vec{x}} - 2Ke^2\int d^2x' G(x,x')\delta^2(x')A(\vec{x'}) = e^{\iota\vec{k}\cdot\vec{x}} - 2Ke^2 G(x,0)A(0), \tag{83}$$

where $G(x,x')$ is the Green function

$$G(x,x') = \int \frac{d^2k}{(2\pi)^2}\frac{e^{\iota k\cdot(x-x')}}{k^2-\omega^2-\iota\epsilon} \tag{84}$$

Using this form for $G(x,x')$ we get

$$A(0) = \frac{1}{1+2Ke^2G(0,0)} \tag{85}$$

where $G(0,0)$ is divergent $\sim (1/2\pi)\ln(\Lambda/\omega)$ (Λ is a cutoff), the divergence arising again due to the fact that we have assumed zero thickness for the string. If we take into account that fact, we get a short range but non-singular potential in place of delta function. We take the above equation to mean

$$A(0) = \frac{1}{1+Ke^2\ln(\Lambda/\omega)/\pi} \tag{86}$$

with a cut-off Λ that depends on the structure of the string. The electric field at the position of the string is proportional to $A(0)$. Since we took the incident wave to be $e^{\iota k \cdot x}$, the incident wave correspond to $A(0) = 1$. The fields induced by the currents excited on the string reduce $A(0)$ by a factor

$$\eta = \frac{1}{1 + Ke^2 \ln(\Lambda/\omega)/\pi} \tag{87}$$

One can see immediately that as $\Lambda \to \infty$, $\eta \to 0$ and there are no currents excited on the string and also no scattered wave. However for realistic values of Λ, η is not extremely small. For example if we take the value $\Lambda \sim 10^{19}$ GeV (corresponding to a string of thickness about Planck length) and $\omega \sim 10^{-10} y^{-1}$ (since we cannot observe processes with longer duration than the present age of the universe), then $\eta \sim 0.6$.

Now we can write the scattering solution as

$$A(x) = e^{\iota k \cdot x} - 2Ke^2 \eta G(x, 0) \tag{88}$$

Using the asymptotic form of the Green function in two dimensions

$$G(x, 0) \stackrel{|x| \to \infty}{\longrightarrow} \sqrt{\iota/8\pi\omega|x|} e^{\iota\omega|x|} \tag{89}$$

we get the scattering amplitude as

$$f = -2Ke^2 \eta \sqrt{\frac{\iota}{8\pi\omega}} \tag{90}$$

As $K \sim 1/\tilde{\lambda}$ and $\ln(\Lambda/\omega) >> 1$, we can approximate η by

$$\eta = \frac{\pi}{Ke^2 \ln(\Lambda/\omega)} \tag{91}$$

This gives, finally, the total cross section per unit length, integrated over scattering angles, as

$$\frac{d\sigma}{dz} = \frac{\pi}{2(\ln(\Lambda/\omega)^2} \lambda \tag{92}$$

where $\lambda = 2\pi/\omega$ is the wavelength of the incident radiation.

This is an interesting result. It shows that, apart from a slowly varying logarithm, the size of the string as measured by the scattering cross section, is independent of its physical dimension. This means that if the string is probed by visible light, it appears to have a thickness of a few tenths of an angstrom. If it is probed by a light of wavelength 30000 light years (galctic dimension) it appears to have a thickness of a few light years.

Suggested Readings

We have not given any specific references in the text. Rather, we have tried to make the thesis as much self contained as possible. However we are adding a short list of references where further references to the original literature may be found.

Text Books

Two recent books which discuss topological defects are:

1. E. Kolb and M.S. Turner, "The Early Universe", Addison Wesley, 1988

2. Collins, et al , "Cosmology and Particle Physics".

Lecture notes

The following lecture notes by some of the contributors to the field are very helpful for beginers.

1. Niel Turok, "Cosmic Strings", lectures presented at the CCAST symposium on Particle Physics and Cosmology, Nanjiing, China 30 June - 13 July 1988. Published in: Fang and Lee (ed), "Cosmology and Particle Physics".

2. A. Vilenkin, "Cosmic Strings and other Topological Defects", lectures presented at the 8^{th} Kyoto Summer Institute, 1985. Published in: Sato and Inami (ed), "Quantum Gravity and Cosmology".

Papers

1. A. Vilenkin, Phys. Rep 121 (1985) 263 . Although old, it is still a good source of information on cosmic strings and domain walls.

2. T.W.B. Kibble, "Cosmic Strings: Current Status", Preprint IMPERIAL/TP/91 92/3.

3. T.W.B. Kibble, J. Phys. A9 (1976) 1387.

4. E. Witten, " Superconducting strings", Nucl.Phys. B249 (1985) 557. The idea of superconducting cosmic strings was first introduced in this paper. It is perhaps also the most readable introduction to the subject.

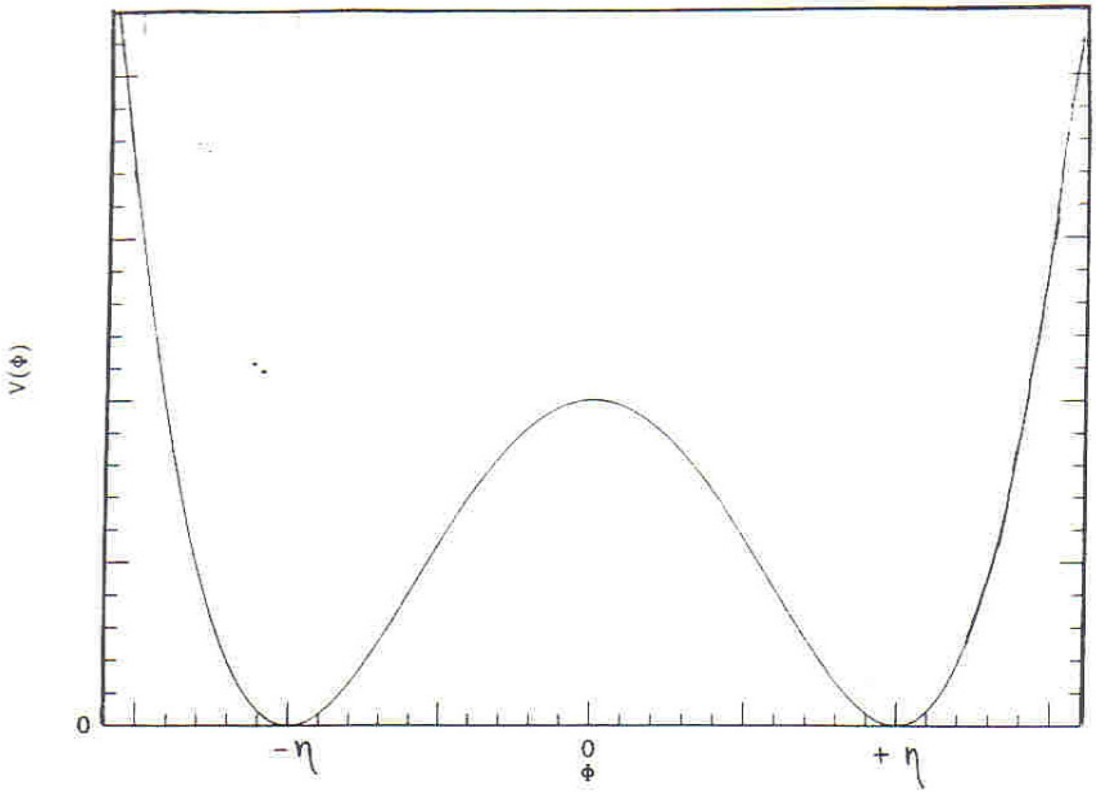

Figure 1: The potential $V(\phi)$

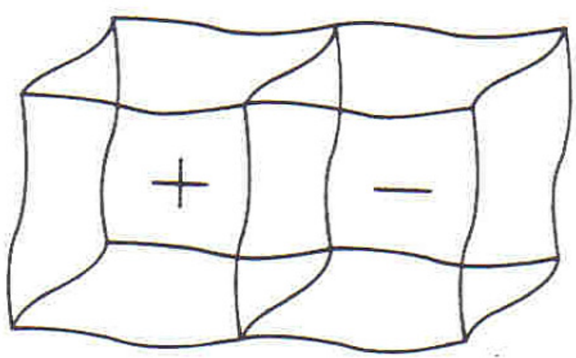

Figure 2: Domain Wall

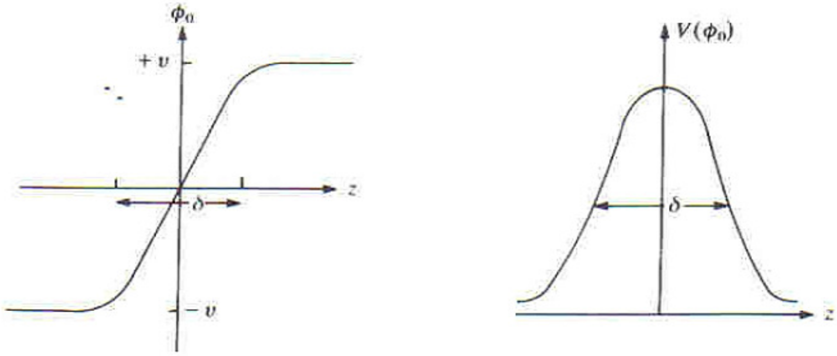

Figure 3: The solution for the Higgs field

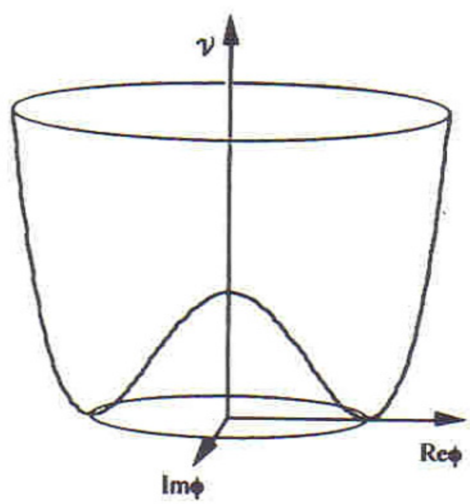

Figure 4: Mexican hat potential

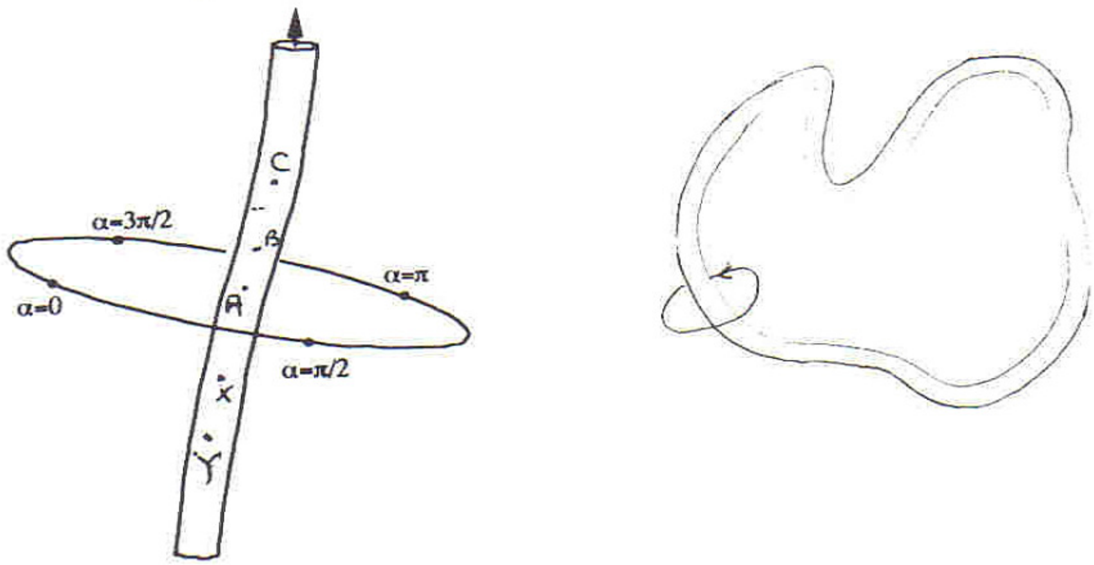

Figure 5: Strings (a) infinite string (b) finite string loop

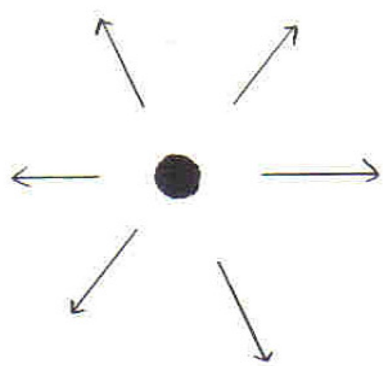

Figure 6: A monopole

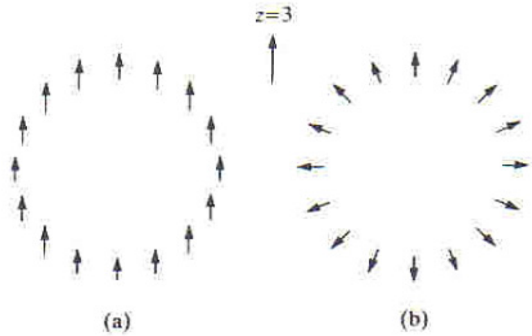

Figure 7: (a) The trivial solution (b) The hedgehog solution

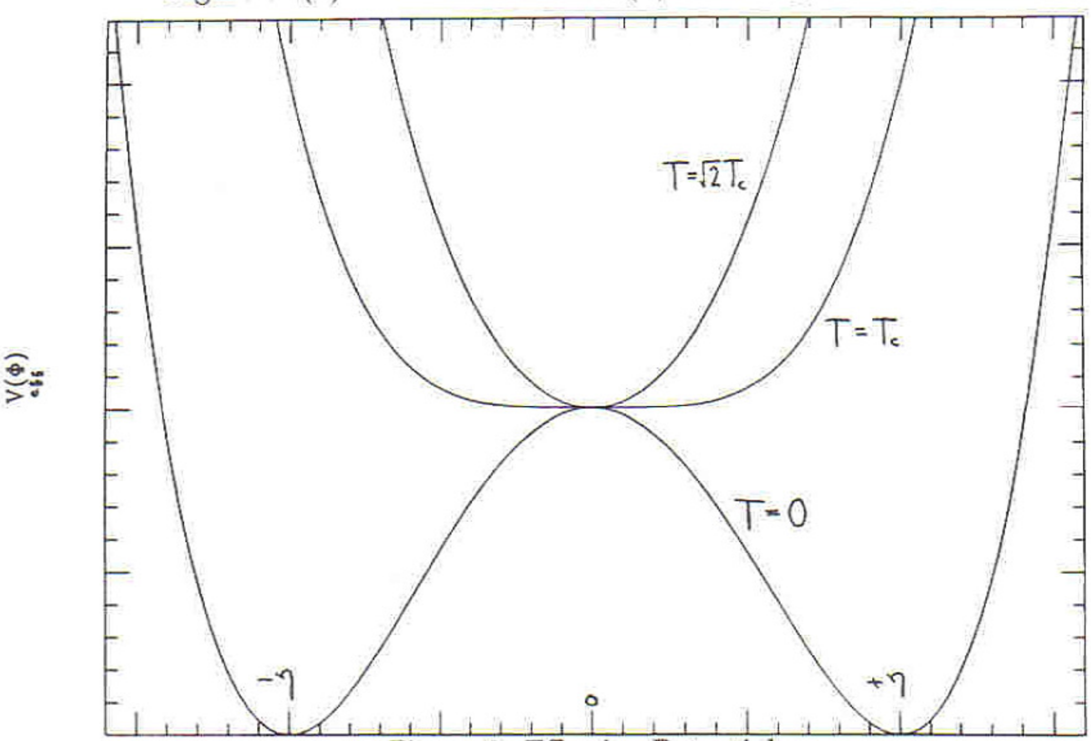

Figure 8: Effective Potential

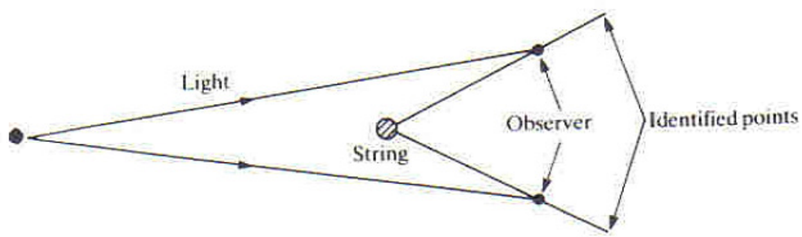

Figure 9: Space around a string